J. Fluid Mech. (1996), *vol.* 319, *pp.* 109–123

Asymptotic expansions in the problem of a solitary wave

By E. A. KARABUT

Lavrentyev Institute of Hydrodynamics, Novosibirsk, 630090, Russia

(Received 26 April 1994 and in revised form 30 August 1995)

There are a number of papers devoted to the construction of the exact solitary wave solution using a series. Power series in amplitude or Fourier series have usually been used. In the present paper we accomplish the exact summation of the Witting (1975) series and show that this series describes other flows, not solitary waves. One such flow is fluid suction under a curvilinear roof. The left half of it is similar to the left half of a maximal-amplitude solitary wave flow.

1. Introduction

The solitary wave at a fluid surface has been the object of study of many authors from its discovery by Russel (1838). Boussinesq (1871), Rayleigh (1876), Korteweg & de Vries (1895) were the pioneer investigators of this problem. They developed approximate theories representing small-amplitude solitary waves.

The exact statement of the problem for a large-amplitude wave requires a more complete theory although many papers have been written on this subject. The existence of the solution was not proved until the results of Lavrentyev (1946) and Friedrichs & Hyers (1954). Stokes (1880) conjectured that the highest wave would be characterized by sharp crest with the angle 120°. The hypothesis was justified by Toland (1978) and Plotnikov (1983). The occurrence of two completely different solitary waves moving with equal velocities has been revealed numerically by Longuet-Higgins & Fenton (1974) and Byatt-Smith & Longuet-Higgins (1976). More recently Plotnikov (1991) proved the non-uniqueness theorem for the solitary wave problem. Craig & Sternberg (1988) proved the symmetry of a solitary wave profile relative to the vertical axis.

Nontheless these and other excellent precise results do not provide methods for determining the free-surface shape, mass, energy and momentum. To evaluate the profile and parameters of the flow two groups of numerical methods were used. The first group is based on a numerical summation of various series representing the solution. Power series with respect to the small wave height were used in the studies of Fenton (1972), Longuet-Higgins & Fenton (1974), Pennel & Su (1984), Pennel (1987). Longuet-Higgins & Fenton carried out particularly extensive calculations. Using the changes of variables in power series and Padé summation method, they estimated many parameters of solitary waves everywhere up to a maximum amplitude. Some other types of series were used by Witting (1975), Ovsyannikov (1991) and Karabut (1994), who used expansions with respect to special families of functions. The numerical summation of different series provides either the same or closely similar results. These results as a whole agree closely with those obtained by numerical methods of the second group involving finite difference approximations applied to the integro-differential equation equivalent to the problem considered. The results are presented in the papers of Byatt-Smith & Longuet-Higgins (1976), Williams (1981), Witting (1981) and Hunter & Vanden-Broeck (1983). There are minor differences in the calculated values of parameters for high-amplitude waves. For example, the value of a ratio of maximum amplitude to a fluid depth at infinity obtained from numerical calculations based on a series method is

$$0.827. \tag{1.1}$$

This number slightly differs from the value

$$0.833, \tag{1.2}$$

obtained by methods of the second group.

Previously, differences in numerical results obtained by different techniques were explained by the unsufficient accuracy of the Padé summation method (Pennel 1987) or by the small number of series terms (Pennel & Su 1984). Witting (1975) explained the differences by the fact that the series used are incomplete. In our opinion this explanation appears to be valid. In the present work further evidence in favour of this assertion is found. The essential result of the present paper is the exact summation of the Witting series. The summation problem is reduced to finding the solution of a special system of ordinary differential equations. It is shown that the Witting series do not describe solitary waves but correspond to some other stationary flows of a heavy fluid with a free boundary. One such solution corresponding to fluid suction under a curvilinear roof is analysed in this paper.

Previously, Villat (1915) found the solution of the problem of flow over an uneven bottom of a specific shape. The exact solution to the problem of fluid flow over the bottom of a step form was gived by Richardson (1920). Additional examples of exact solutions (their number is no more than 10) can be found in Gurevich (1965). In this paper we added a new one-parameter family of exact solutions to these separate examples.

Two theories of surface waves, namely the linear theory of small-amplitude waves and the nonlinear theory of long waves, are well developed. The former does not describe solitary waves and the latter describes only small-amplitude solitary waves. In the theory suggested by Davies (1951) the initial nonlinear boundary value problem is changed slightly so that the new problem admits the exact solution. This solution may be considered as the first approximation. The application of this approach to the solitary wave problem was realized by Packham (1952). The theory describes surface waves over all range of amplitudes with an accuracy of 11–13%, and in Goody & Davies (1957) this solution is tabulated.

Another theory may be developed using the results of this article. The flows described by the Witting series are very close to solitary waves. This permits one to use these flows for an approximate modelling of solitary waves. The flow considered in this paper is close to the maximum-amplitude solitary wave.

2. Formulation of the problem

Consider the two-dimensional steady irrotational flow under gravity of an inviscid incompressible fluid of uniform density over an even bottom. The capillarity effects are neglected. Let the Cartesian coordinate X-axis be directed along the bottom and the Y-axis be directed vertically upward (figure 1a). The origin of the coordinate

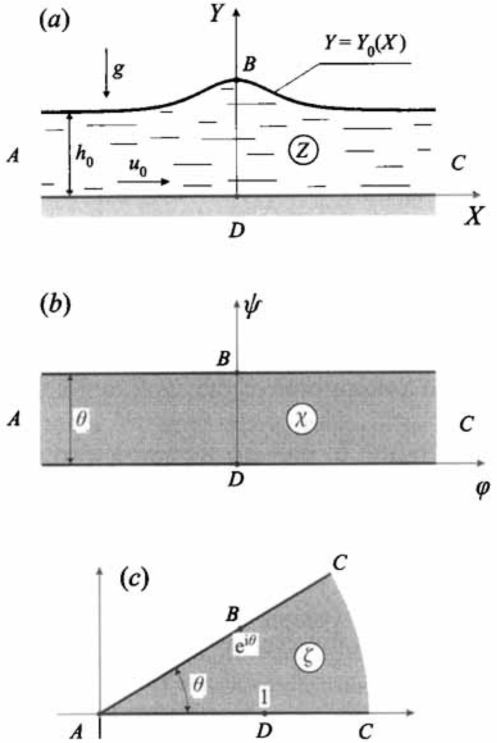

FIGURE 1. (*a*) Z-plane; (*b*) χ-plane; (*c*) ζ-plane.

system is on the bottom and the Y-axis crosses the free surface at the highest point. The velocity of the flow and the depth of the fluid at infinity are denoted as u_0 ($u_0 > 0$) and h_0 respectively, g is the acceleration due to gravity.

To describe a solitary wave we have to find a solution of the Euler equations with a local elevation of the free boundary $Y = Y_0(X)$, diminishing at infinity

$$\lim_{|X|\to\infty} Y_0(X) = h_0.$$

The solution depends on one parameter. The Froude number

$$Fr = \frac{u_0}{(gh_0)^{1/2}} > 1$$

or the Stokes parameter θ may be considered. The value of θ may be determined from

$$Fr^2 = \frac{\tan\theta}{\theta}.$$

This equation has infinitely many roots. The Stokes parameter is the root located in the interval $0 \leqslant \theta < \pi/2$. The limit $\theta \to 0$ corresponds to small-amplitude solitary waves. The amplitude increases with growing θ.

In some earlier papers the solution of the problem of a solitary wave was represented as the following series:

$$\frac{Y_0(X)}{h_0} = 1 + \sum_{j=1}^{\infty} \theta^{2j} \sum_{n=1}^{j} \alpha_{jn} \left(\operatorname{sech} \frac{\theta X}{2h_0} \right)^{2n}, \tag{2.1}$$

$$\frac{Y_0(X)}{h_0} = 1 + \sum_{n=1}^{\infty} a_n \left(\operatorname{sech} \frac{\theta X}{2h_0} \right)^{2n}. \tag{2.2}$$

These series are similar in appearance and they can be found from each other by interchanging the summation order. However, they are obtained with different assumptions. The asymptotic expansion (2.1) is suited to small amplitudes, whereas when constructing (2.2) the assumption of the small amplitude is not used.

The series (2.1) is a classical series of the shallow water theory. Friedrichs (1948) proposed a systematic procedure to find the highest approximations of shallow water theory. The first, second and third approximations were obtained by Keller (1948), Laitone (1960) and Grimshaw (1971), respectively. Using computers the solution of the form (2.1) is found: up to θ^{18} by Fenton (1972), up to θ^{28} by Longuet-Higgins & Fenton (1974), up to θ^{34} by Pennel & Su (1984), up to θ^{54} by Pennel (1987).

The series (2.2) was used by Pennel & Su (1984). Substituting (2.2) into the boundary conditions, one can find the coefficients a_n. The first coefficient a_1 remains indefinite and all the other ones are determined by it.

We shall formulate a solitary wave problem in the plane of the complex potential $\Phi + \mathrm{i}\Psi$ and shall consider similarities of the series (2.1), (2.2). The problem is simplified because the boundary-value problem in a unknown region is replaced by the boundary-value problem in the fixed infinite strip in the plane $\Phi + \mathrm{i}\Psi$. Let the velocity potential Φ take a zero value on the symmetry line BD (figure 1a). The stream function Ψ is chosen so that it takes a zero value at the bottom.

In the plane of the non-dimensional complex potential

$$\chi = \varphi + \mathrm{i}\psi = \frac{\theta}{h_0 u_0}(\Phi + \mathrm{i}\Psi)$$

the flow region occupies the infinite strip of width θ (figure 1b). The problem is to find a conformal mapping of this strip on the flow region in the physical plane. Let this conformal mapping be represented in the form

$$Z = X + \mathrm{i}Y = \frac{h_0}{\theta}\Big(\chi + W(\chi) \Big). \tag{2.3}$$

The problem is reduced to determining the function

$$W(\chi) = A(\varphi, \psi) + \mathrm{i}B(\varphi, \psi),$$

which is analytical in the strip

$$0 < \psi < \theta, \quad -\infty < \varphi < \infty$$

and satisfies the following boundary conditions. At the upper strip boundary the constant-pressure condition (the Bernoulli integral) is fulfilled. It can be written in the form proposed by Ovsyannikov (1991):

$$B_\psi - \nu B = f \qquad (\psi = \theta), \tag{2.4}$$

$$f = \frac{2\nu^2 B^2}{1 - 2\nu B} - \tfrac{1}{2}\left(B_\varphi^2 + B_\psi^2\right), \quad \nu = \cot\theta.$$

At the lower strip boundary the flat-bottom condition is fulfilled:

$$B = 0 \qquad (\psi = 0). \tag{2.5}$$

The additional condition

$$\lim_{|\varphi|\to\infty} B(\varphi, \psi) = 0, \tag{2.6}$$

distinguishes the solitary wave solution from another solutions of the surface waves theory. In the plane of the complex potential series similar to (2.1) and (2.2) have the following form respectively:

$$W = \sum_{j=1}^{\infty} \theta^{2j} \sum_{n=1}^{j} \beta_{jn} \int_0^{\chi} \mu^n \mathrm{d}\chi, \tag{2.7}$$

$$W = \sum_{n=1}^{\infty} b_n \int_0^{\chi} \mu^n \mathrm{d}\chi. \tag{2.8}$$

Here

$$\mu = \frac{1}{2}\left(\operatorname{sech}\frac{\chi}{2}\right)^2.$$

The series (2.8) was proposed by Ovsyannikov (1991). One can obtain recurrent formulae for sequentially found coefficients b_n by substituting (2.8) into the boundary condition (2.4). The coefficient b_1 remains undefined and the rest are functions of it. The problem of determining the first coefficient was solved by Ovsyannikov. He suggested determining b_1 from the equation

$$\int_0^{\infty} f \cosh\varphi \,\mathrm{d}\varphi - b_1\left(\frac{\theta}{\sin\theta} - \cos\theta\right) = 0. \tag{2.9}$$

Here f is the function from the boundary condition (2.4). The numerical solution of the equation (2.9) was performed by Karabut (1994).

Witting (1975) proposed considering the power series with respect to $\mathrm{e}^{\lambda\chi}$ $(\lambda > 0)$:

$$W = \sum_{j=1}^{\infty} E_j \mathrm{e}^{j\lambda\chi}. \tag{2.10}$$

Let E_j be real numbers. Then the condition on the bottom (2.5) will be fulfilled. The boundary condition of pressure constancy (2.4) may also be satisfied. For this purpose we substitute the imaginary part of (2.10)

$$B = \sum_{j=1}^{\infty} E_j \mathrm{e}^{j\lambda\varphi} \sin j\lambda\psi$$

into (2.4) and collect the terms with the same exponents $e^{j\lambda\varphi}$. As a result we obtain the recurrent formulae for coefficients E_j:

$$(j\lambda \cos j\lambda\theta - \nu \sin j\lambda\theta) E_j = R_j(E_1, E_2, \ldots, E_{j-1}; \theta, \lambda). \tag{2.11}$$

For $j = 1$ we obtain the equation

$$(\lambda \cos \lambda\theta - \nu \sin \lambda\theta) E_1 = 0.$$

Hence the coefficient E_1 is undefined and λ should be found from the equation

$$\tan \lambda\theta = \lambda \tan \theta. \tag{2.12}$$

Using the designation $\tilde{\theta} = \lambda\theta$, this equation may be rewritten in the form

$$\frac{\tan \tilde{\theta}}{\tilde{\theta}} = \frac{\tan \theta}{\theta} = Fr^2$$

or in the form of the system:

$$\eta = \tan \tilde{\theta}, \quad \eta = Fr^2 \tilde{\theta},$$

the graphical solution of which is given in figure 2. There is an infinite set of roots

$$\theta, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \dots,$$

which allow us to obtain an infinite set of solutions to equation (2.12)

$$\lambda_0 = 1, \lambda_1, \lambda_2, \lambda_3, \dots .$$

The following question arises: what harmonic λ_j have we to choose? Witting (1975) suggested that the solution of the solitary wave problem should include all harmonics λ_j. None of the above series (2.1), (2.2), (2.7), (2.8) complies with the requirement. At the same time he proposed that the harmonic $\lambda_0 = 1$ contributes more significantly to the solution than other harmonics. Therefore, the solution of the solitary wave problem may be approximated as the series (2.10), setting $\lambda = 1$:

$$W = \sum_{j=1}^{\infty} E_j \zeta^j \qquad \left(\zeta = \mathrm{e}^{\chi},\ \mathrm{Im}\, E_j = 0\right). \tag{2.13}$$

Witting computed more than 200 terms of this series and then numerically summed it. For $\theta = \pi/4$ he constructed a free surface and for $\theta = \pi/3$ he found a singular point at the free surface. There are singularities at the free surface only in the case of a limiting amplitude wave. Therefore, the conclusion was drawn that the solitary wave of a limiting amplitude corresponds to $\theta \approx \pi/3$.

Below is shown how the exact summation of the series (2.13) may be done and this is exemplified for $\theta = \pi/3$.

3. The Witting series

The boundary value problem (2.4), (2.5), (2.6) is invariant under the translation $\varphi \to \varphi + \varphi_0$. The solution of this problem in the form of power series (2.13) also does not depend on φ_0. Therefore the recurrent formulae (2.11) with $\lambda = 1$ should be invariant under the transformation

$$E_j \to \delta^j E_j, \quad \delta = \mathrm{e}^{-\varphi_0}, \quad j \geqslant 1.$$

Let $\delta = 1/E_1$. Then the values

$$\tilde{E}_j = \delta^j E_j = E_j / E_1^j,$$

defined by the recurrent formulae

$$(j \cos j\theta - \nu \sin j\theta)\tilde{E}_j = R_j\left(1, \tilde{E}_2, \dots, \tilde{E}_{j-1}; \theta, 1\right), \quad j \geqslant 2$$

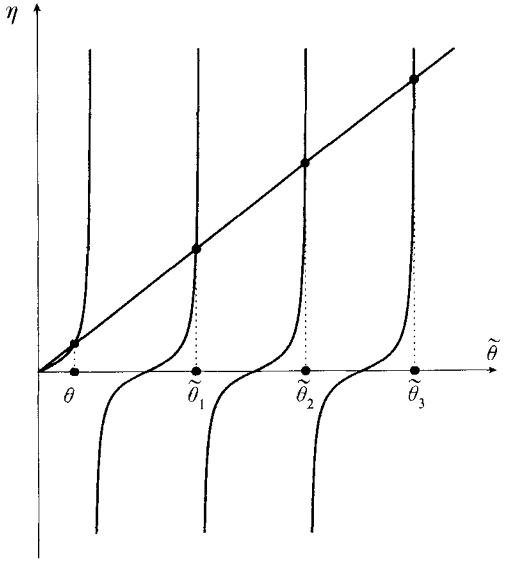

FIGURE 2. Graphical illustration of equation (2.12).

are the functions of θ alone, i.e. they are independent of E_1. It follows that

$$E_j(v, E_1) = E_1^j \tilde{E}_j(v) \qquad (j \geqslant 2). \tag{3.1}$$

Consider, for example, the first three formulae of (3.1)

$$\begin{aligned} E_2 &= E_1^2 \left(-\tfrac{3}{4}v^2 + \tfrac{1}{4}\right), \\ E_3 &= E_1^3 \left(\tfrac{9}{16}v^4 - \tfrac{7}{8}v^2 + \tfrac{1}{16}\right), \\ E_4 &= E_1^4 \left(-\tfrac{135}{64}v^8 + \tfrac{261}{32}v^6 - \tfrac{41}{8}v^4 + \tfrac{19}{32}v^2 - \tfrac{1}{64}\right) / \left(5v^2 - 1\right). \end{aligned}$$

The first coefficient E_1 is undefined, however it may be taken as an arbitrary positive number. In figure 1(c) the picture in the plane $\zeta = \mathrm{e}^\chi$ is presented. The ray emerging from an origin of the coordinate system and inclined at the angle θ to a real axis corresponds to a free surface. Therefore substituting $\zeta = r\mathrm{e}^{\mathrm{i}\theta}$ in (2.13) we get the free-surface profile

$$Z = X + \mathrm{i}Y = \frac{h_0}{\theta}\left(\ln r + \mathrm{i}\theta + \sum_{j=1}^{\infty} \tilde{E}_j \mathrm{e}^{\mathrm{i}\theta j}(E_1 r)^j\right).$$

Introduce $s = E_1 r$. We get another parametric representation

$$Z(s, E_1) = \frac{h_0}{\theta}\left(\ln s - \ln E_1 + \mathrm{i}\theta + F(s)\right),$$

$$F(s) = \sum_{j=1}^{\infty} \tilde{E}_j \mathrm{e}^{\mathrm{i}\theta j} s^j \qquad (s \geqslant 0).$$

It follows from these formulae that the variation of E_1 results in the translation of the profile along the X-axis. Thus, the Witting solution with fixed θ describes a unique flow with an accuracy of a translation along X-axis.

Let $\omega = \mathrm{e}^{\mathrm{i}\theta}$. From this point on we shall consider W as a function of the variable $\zeta = \mathrm{e}^{\chi}$. Consider the infinite set of functions:

$$\left.\begin{aligned} P_1(\zeta) &= W(\zeta),\\ P_2(\zeta) &= W\left(\zeta\omega^2\right),\\ &\vdots\\ P_l(\zeta) &= W\left(\zeta\omega^{2l-2}\right),\\ &\vdots \end{aligned}\right\} \tag{3.2}$$

This set represents the function $W(\zeta)$ in various coordinate systems having a common centre at point A (figure 1c) and rotated at an angle 2θ with respect to each other. In general, the function $W(\zeta)$ at point A may have a branch point singularity. It may happen that the initial value of $W(\zeta)$ would be never attained after any finite number of rotations about A. The equality $P_l \equiv P_j$ $(l \neq j)$ will not be fulfilled even in the case of rational number θ/π.

Suppose that there exists a ring $0 \leqslant r_1 < |\zeta| < r_2$ which has no singular points of the function $W(\zeta)$. Let us show that the problem of determining $W(\zeta)$ satisfying the conditions (2.4), (2.5) in this ring is equivalent to finding of the solution of the infinite system of ordinary differential equations for the unknown functions $P_l(\zeta)$.

Rewrite the boundary-value condition (2.4) in the form

$$\left|\frac{\mathrm{d}(W+\chi)}{\mathrm{d}\chi}\right|^2 = \frac{1}{1-2\nu\,\mathrm{Im}\,W}, \quad \nu = \cot\theta \quad (\psi = \theta). \tag{3.3}$$

Differentiating along the free surface we obtain the equality

$$\left(r\frac{\mathrm{d}W\left(r\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{d}r}+1\right)\left(r\frac{\mathrm{d}\overline{W\left(r\mathrm{e}^{\mathrm{i}\theta}\right)}}{\mathrm{d}r}+1\right) = \frac{1}{1+\mathrm{i}\nu\left[W\left(r\mathrm{e}^{\mathrm{i}\theta}\right)-\overline{W\left(r\mathrm{e}^{\mathrm{i}\theta}\right)}\right]}. \tag{3.4}$$

In the ζ-plane the bottom condition is fulfilled on the positive real axis:

$$\mathrm{Im}\,W = 0.$$

Therefore, the function $W(\zeta)$ defined in the upper half-plane admits analytic continuation into the lower half-plane according to the symmetry principle:

$$\overline{W\left(r\mathrm{e}^{\mathrm{i}\theta}\right)} = W\left(r\mathrm{e}^{-\mathrm{i}\theta}\right). \tag{3.5}$$

We have from (3.2)

$$P_l\left(r\mathrm{e}^{-\mathrm{i}\theta(2l-1)}\right) = W\left(r\mathrm{e}^{-\mathrm{i}\theta}\right),$$
$$P_{l+1}\left(r\mathrm{e}^{-\mathrm{i}\theta(2l-1)}\right) = W\left(r\mathrm{e}^{\mathrm{i}\theta}\right).$$

Referring to (3.5) we can conclude that the functions $P_l(\zeta)$, $P_{l+1}(\zeta)$, taken at the ray $\zeta = r\mathrm{e}^{-\mathrm{i}\theta(2l-1)}$ are no more nor less then the functions $\overline{W}$, W, respectively taken at the free surface ABC. Thus, (3.4) may be rewritten:

$$\left(r\frac{\mathrm{d}P_{l+1}\left(re^{-\mathrm{i}\theta(2l-1)}\right)}{\mathrm{d}r}+1\right)\left(r\frac{\mathrm{d}P_{l}\left(re^{-\mathrm{i}\theta(2l-1)}\right)}{\mathrm{d}r}+1\right)$$
$$=\frac{1}{1+\mathrm{i}\nu\left[P_{l+1}\left(re^{-\mathrm{i}\theta(2l-1)}\right)-P_{l}\left(re^{-\mathrm{i}\theta(2l-1)}\right)\right]}.$$

Here the derivatives do not need to be taken along the given ray if we take into account the analyticity of the functions P_l, P_{l+1}. Thus, the last equation may be rewritten as

$$\left(\zeta\frac{\mathrm{d}P_{l+1}}{\mathrm{d}\zeta}+1\right)\left(\zeta\frac{\mathrm{d}P_{l}}{\mathrm{d}\zeta}+1\right)=\frac{1}{1+\mathrm{i}\nu(P_{l+1}-P_l)}\qquad\left(\zeta=re^{-\mathrm{i}\theta(2l-1)}\right).$$

We analytically continued the differential equation from the ray $\zeta = re^{-\mathrm{i}\theta(2l-1)}$ to the ring. Then the functions P_l have to satisfy the following infinite system of ordinary differential equations:

$$\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi}+1\right)\left(\frac{\mathrm{d}P_1}{\mathrm{d}\chi}+1\right)=\frac{1}{1+\mathrm{i}\nu(P_2-P_1)},$$
$$\left(\frac{\mathrm{d}P_3}{\mathrm{d}\chi}+1\right)\left(\frac{\mathrm{d}P_2}{\mathrm{d}\chi}+1\right)=\frac{1}{1+\mathrm{i}\nu(P_3-P_2)},$$
$$\vdots$$

Firstly, if we assume that $W(\zeta)$ is the analytical function in the vicinity $\zeta = 0$ and, secondly, that $\theta = \pi m/n$ (m, n are the integers) then this system will be finite. In this case $P_{n+1} \equiv P_1$. Note that the assumption of analyticity is connected with the form of the Witting solution (2.13) which is a power series with respect to the variable ζ.

Hence if θ/π is a rational number then for summing (2.13) it is sufficient to find

$$P_1,\ P_2,\ \ldots,\ P_n,$$

i.e. the solution of the system of n ordinary differential equations

$$\left(\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\chi}+1\right)\left(\frac{\mathrm{d}P_j}{\mathrm{d}\chi}+1\right)=\frac{1}{1+\mathrm{i}\nu(P_{j+1}-P_j)},\tag{3.6}$$
$$P_{n+1}\equiv P_1,\qquad j=1,\ldots,n$$

satisfying, as is evident from (3.2), the conditions

$$P_j = E_1\zeta\omega^{2j-2}+O(\zeta^2);\quad \zeta=e^{\chi},\quad \omega=e^{\mathrm{i}\theta};\quad j=1,\ldots,n$$

and then to determine W by the equality $W = P_1$.

4. A special case

Here, we shall use the Witting series (2.13) for an approximate description of a solitary wave with the limiting amplitude. What value of θ has to be chosen in this case? It is necessary to take the value of θ such that the free surface has singularities. The solution of the system (3.6) has this feature when $\theta = \pi/3$. This agrees with the Witting (1975) numerical results. The numerical results of other authors also give value close to $\pi/3$.

For $\theta = \pi/3$ the system (3.6) consists of only three equations

$$(P_2' + 1)(P_1' + 1) = 1/f_1, \ (P_3' + 1)(P_2' + 1) = 1/f_2, \ (P_1' + 1)(P_3' + 1) = 1/f_3,$$

where

$$f_1 = 1 + \frac{i}{\sqrt{3}}(P_2 - P_1), \ f_2 = 1 + \frac{i}{\sqrt{3}}(P_3 - P_2), \ f_3 = 1 + \frac{i}{\sqrt{3}}(P_1 - P_3). \tag{4.1}$$

This system may be reduced to the normal form

$$P_1' + 1 = \frac{f_2}{(f_1 f_2 f_3)^{1/2}}, \quad P_2' + 1 = \frac{f_3}{(f_1 f_2 f_3)^{1/2}}, \quad P_3' + 1 = \frac{f_1}{(f_1 f_2 f_3)^{1/2}}. \tag{4.2}$$

Introduce the following functions

$$\left.\begin{aligned} S_1 &= E_1\zeta + E_4\zeta^4 + E_7\zeta^7 + E_{10}\zeta^{10} + \dots, \\ S_2 &= E_2\zeta^2 + E_5\zeta^5 + E_8\zeta^8 + E_{11}\zeta^{11} + \dots, \\ S_3 &= E_3\zeta^3 + E_6\zeta^6 + E_9\zeta^9 + E_{12}\zeta^{12} + \dots. \end{aligned}\right\} \tag{4.3}$$

From (3.2) it follows that

$$P_1 = W(\zeta), \ P_2 = W\left(e^{i2\pi/3}\zeta\right), \ P_3 = W\left(e^{i4\pi/3}\zeta\right).$$

Using (4.3) and (2.13), we obtain the relations

$$\left.\begin{aligned} P_1 &= S_1 + S_2 + S_3, \\ P_2 &= e^{2i\pi/3}S_1 + e^{4i\pi/3}S_2 + S_3, \\ P_3 &= e^{4i\pi/3}S_1 + e^{2i\pi/3}S_2 + S_3. \end{aligned}\right\} \tag{4.4}$$

Let us transform the system (4.2) to the new unknowns S_1, S_2, S_3. It follows from (4.4) and (4.1) that f_1, f_2, f_3 do not depend on S_3. The linear combination of the equations gives the equalities

$$S_1' = \frac{S_1}{(f_1 f_2 f_3)^{1/2}}, \quad S_2' = -\frac{S_2}{(f_1 f_2 f_3)^{1/2}}.$$

Hence S_1 and S_2 satisfy the relation

$$S_1 \, S_2 = \text{const.}$$

Here, the constant is equal to zero because it follows from (4.3) that $S_1 = S_2 = 0$ at $\zeta = 0$. Thus one of these functions is the identical null. Since $E_1 > 0$, this is S_2. Taking it into account we have $f_1 f_2 f_3 = 1 + S_1^3$. As a consequence

$$\frac{dS_1}{d\chi} = \frac{S_1}{\left(1 + S_1^3\right)^{1/2}}, \qquad \frac{d(S_3 + \chi)}{d\chi} = \frac{1}{\left(1 + S_1^3\right)^{1/2}}. \tag{4.5}$$

Here, the radical branch is chosen so that $\sqrt{1} = 1$ as $\zeta \to 0$. If the functions $S_1(\chi), S_3(\chi)$ are known, the Witting solution may be expressed in the form

$$W = S_1 + S_3.$$

From (4.5) it follows that

$$\frac{d(S_3 + \chi)}{dS_1} = \frac{1}{S_1}.$$

Integrating it, we obtain the relation

$$S_3 + \chi = \log S_1 + \text{const}.$$

We can find the integration constant by considering the asymptotics at $\zeta \to 0$. This gives the result

$$\chi + W = S_1 + \log \frac{S_1}{E_1}. \tag{4.6}$$

The solution of the first of equations (4.5) has singularities at the points where $1 + S_1^3 = 0$. We shall show that one of these singularities lies on the ray $\zeta = re^{i\pi/3}$ corresponding to the free surface. From (4.3) it follows that

$$S_1 = e^{i\pi/3} Q \tag{4.7}$$

on this ray. Here Q is the real-valued function in the vicinity of the point $\zeta = 0$ defined by the series

$$Q = E_1 r - E_4 r^4 + E_7 r^7 - \dots .$$

But in general Q is not a real-valued function. According to (4.5) this function satisfies

$$\frac{dQ}{d\varphi} = \frac{Q}{\left(1 - Q^3\right)^{1/2}} \qquad (\psi = \pi/3). \tag{4.8}$$

When φ varies from $-\infty$ to a certain value φ^* the function Q monotonically increases from 0 to 1. The function Q cannot be real–valued for $\varphi > \varphi^*$. Hence $\chi^* = \varphi^* + i\pi/3$ is a singular point.

Let us find φ^*. Integrating (4.8) gives

$$3\varphi + \text{const} = 2\left(1 - Q^3\right)^{1/2} + \log \frac{1 - \left(1 - Q^3\right)^{1/2}}{1 + \left(1 - Q^3\right)^{1/2}}. \tag{4.9}$$

The integration constant is defined by the asymptotics at $\varphi = \log r \to -\infty$. Next inserting $Q^3 = 1$ into (4.9), we have

$$\varphi^* = \ln(\delta/E_1), \qquad \delta = \left(4/e^2\right)^{1/3} \approx 0.815. \tag{4.10}$$

Let us introduce the auxiliary function

$$u(\chi) = S_1^3. \tag{4.11}$$

From (4.5) it follows that

$$d\chi = \frac{1}{3} du \frac{(1 + u)^{1/2}}{u}.$$

Integration of this equation gives the result

$$\chi = \frac{1}{3} \int_{-1}^{u} d\xi \frac{(1 + \xi)^{1/2}}{\xi} + \varphi^* + i\frac{\pi}{3}. \tag{4.12}$$

The formulae (4.6), (4.10), (4.11), (4.12) completely describe the Witting solution for $\theta = \pi/3$. Note that (4.12) is the Schwarz–Christoffel formula for the conformal mapping of the upper half-plane u shown in figure 3(a), onto a polygon in the plane χ, shown in figure 3(b). The function (4.11) gives the conformal mapping of the upper half-plane u onto a wedge of interior angle $\pi/3$ in the plane S_1 (figure 3c). The side AB of the polygon corresponds to the upper boundary of the strip, wherein the initial

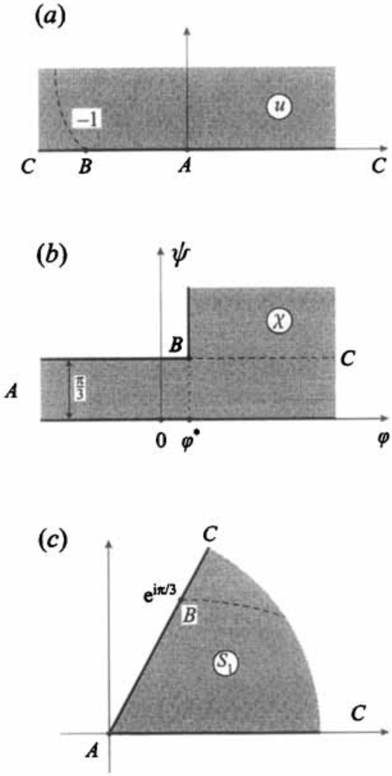

FIGURE 3. (*a*) u-plane; (*b*) χ-plane; (*c*) S_1-plane.

boundary-value problem is considered. Let us show that the condition (3.3) is fulfilled on AB. Substituting of (4.7) into (4.6) gives

$$\chi + W = e^{i\pi/3} Q + i\frac{\pi}{3} + \log\frac{Q}{E_1} \qquad (0 \leqslant Q \leqslant 1). \tag{4.13}$$

Taking into account (4.8), we have

$$\left|\frac{d(W+\chi)}{d\chi}\right|^2 = \left(e^{i\pi/3} + \frac{1}{Q}\right)\left(e^{-i\pi/3} + \frac{1}{Q}\right)\frac{Q^2}{1-Q^3} = \frac{1}{1-Q}.$$

From (4.13) it follows that

$$\operatorname{Im} W = \sin(\tfrac{1}{3}\pi)Q.$$

Therefore we see that

$$\frac{1}{1-2\nu \operatorname{Im} W} = \frac{1}{1-Q}.$$

Thus the constant-pressure condition (3.3) is fulfilled on AB and this boundary section corresponds to the free surface.

The point B is singular. In its vicinity

$$\chi + W \sim (\chi - \chi^*)^{2/3}.$$

Therefore the fluid boundary has a cusp B with an angle of 120°.

In the remainder of the upper boundary of the strip BC (shown by the dotted line in figure 3*b*) the constant-pressure condition is not valid. However, BC is the streamline.

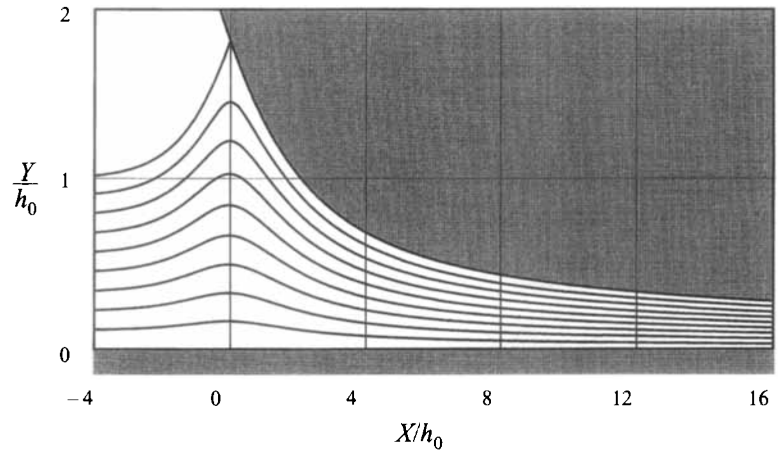

FIGURE 4. A steady flow of depth h_0, passing from the left is sucked into a slit between a rectilinear bottom and a curvilinear roof. The streamlines of the flow are shown.

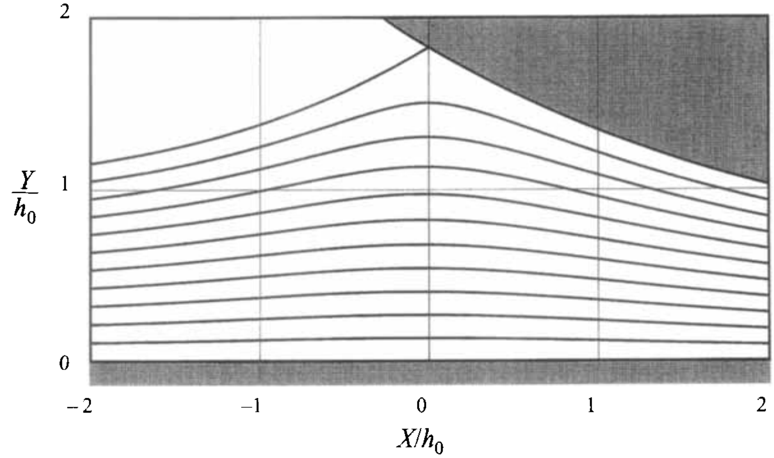

FIGURE 5. The picture of the flow which is close to the maximal-amplitude solitary wave.

Hence, this boundary may be considered as a curvilinear wall. The asymptotic analysis of the solution as $\varphi \to \infty$ ($\psi = \pi/3$) gives the following asymptotic behaviour for this wall:

$$X \sim \varphi^{2/3}, \qquad Y \sim \varphi^{-1/3}.$$

The curvilinear wall approaches the bottom as $X \to \infty$, the velocity of the flow tends to infinity, and the pressure tends to minus infinity. Thus the exact solution of the free boundary problem describing fluid suction under a curvilinear roof is found.

The roof shape is given by rather complicated formulae, conversely the shape of the free surface is given by simple explicit formulae. For the derivation let us insert (4.13) in (2.3) and set $E_1 = \mathrm{e}^{1/2}$. Separating the real and imaginary parts we have

$$\frac{X}{h_0} = \frac{3}{2\pi}(Q - 1 + 2\log Q), \qquad \frac{Y}{h_0} = \frac{3^{3/2}}{2\pi}Q + 1 \qquad (0 \leqslant Q \leqslant 1). \tag{4.14}$$

The streamlines and the roof shape are shown in figure 4. The corresponding picture with equal scales on each axis is given in figure 5.

The solution constructed for $\theta = \pi/3$ is close to the maximal-amplitude solitary wave. This is most evident from figure 5 where the streamlines are near-symmetrical with respect to the vertical axis. Let us drop a perpendicular to the bottom from the contact point of the free surface and a curvilinear roof. A comparison with numerical results shows that the part of the constructed flow located to the left of the perpendicular ($X < 0$) is close to the left half of the maximal-amplitude solitary wave flow. For example, substituting $Q = 1$ in (4.14) one can find the dimensionless amplitude

$$\frac{Y_0(0) - h_0}{h_0} = \frac{3^{3/2}}{2\pi} = 0.82699.$$

This value coincides with that of (1.1) but differs by 0.74% from that of (1.2). Using (4.14) one can calculate the dimensionless doubled mass of the left half of the constructed flow:

$$M = \frac{2}{h_0^2} \int_{-\infty}^{0} (Y_0 - h_0)\mathrm{d}X = \frac{5 \times 3^{5/2}}{4\pi^2} = 1.974.$$

This value differs by 0.2% from the mass of a limiting-amplitude solitary wave of 1.970 numerically calculated by Williams (1981) and Witting (1981). Similarly, the dimensionless doubled potential energy of the left half of the constructed solution

$$V = \frac{2}{g h_0^3} \int_{-\infty}^{0} \mathrm{d}X \int_{h_0}^{Y_0(X)} gY\,\mathrm{d}Y - M = \frac{27}{2\pi^3} = 0.4354$$

differs by 0.5% from the value of 0.4376 calculated for the solitary wave by the same authors. The difference between the constructed solution and the maximal–amplitude solitary wave does not exceed 1%.

The solutions of system (3.6) for other rational $\theta/\pi < 1/3$ are likewise the solutions of the free boundary problems for flows with gravity. These flows are also close to solitary waves of non-maximal amplitude.

The author acknowledges the kind attention and help given by Academician L. V. Ovsyannikov and Professor V. M. Teshukov to this study.

REFERENCES

Boussinesq, J. 1871 Theorie de l'intumescence liquide appelée on de solitaire on de translation se propageant dans un canal rectangulaire. *Comptes Rendus* **72**, 755.

Byatt-Smith, J. G. B. & Longuet-Higgins, M. S. 1976 On the speed and profile of steep solitary waves. *Proc. R. Soc. Lond.* A **350**, 175.

Craig, W. & Sternberg, P. 1988 Symmetry of solitary waves. *Commun. Partial Diffl Equat.* **13**, 603.

Davies, T. V. 1951 The theory of symmetrical gravity waves of finite amplitude I. *Proc. R. Soc. Lond.* A **208**, 475.

Fenton, J. 1972 A ninth-order solution for the solitary wave. *J. Fluid Mech.* **53**, 257.

Friedrichs, K. O. 1948 On the derivation of the shallow water theory. Appendix to the formation of breakers and bores by J. J. Stoker. *Commun. Pure Appl. Maths* **1**, 81.

Friedrichs, K. O. & Hyers, D. H. 1954 The existence of solitary waves. *Commun. Pure Appl. Maths* **7**, 517.

Goody, A. J. & Davies, T. V. 1957 The theory of symmetrical gravity waves of finite amplitude IV. Steady, symmetrical, periodic waves in a channel of finite depth. *Q. J. Mech. Appl. Maths* **10**, 1.

Grimshaw, R. 1971 The solitary wave in water of variable depth. Part 2. *J. Fluid Mech.* **46**, 611.

Gurevich, M. I. 1965 *Theory of Jets in Ideal Fluids*. Academic Press.

HUNTER, J. K. & VANDEN-BROECK, J.-M. 1983 Accurate computations for steep solitary waves. *J. Fluid Mech.* **136**, 63.

KARABUT, E. A. 1994 The numerical analysis of asymptotical representation of solitary waves. *Prikl. Mekh. i Tekh. Fiz.* (5), 44 (in Russian).

KELLER, J. B. 1948 The solitary wave and periodic waves in shallow water. *Commun. Pure Appl. Maths* **1**, 323.

KORTEWEG, D. J. & VRIES, G. DE 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. *Phil. Mag.* (5) **39**, 422.

LAITONE, E. V. 1960 The second approximation to solitary and cnoidal waves. *J. Fluid Mech.* **9**, 430.

LAVRENTYEV, M. A. 1946 On the theory of long waves. *Zb. Praz Inst. Matemat. Akad. Ukr. Nayk.* **8**, 13 (in Ukraine).

LONGUET-HIGGINS, M. S. & FENTON, J. D. 1974 On the mass, momentum, energy and circulation of a solitary wave. II. *Proc. R. Soc. Lond.* A **340**, 471.

OVSYANNIKOV, L. V. 1991 On an asymptotic representation of solitary waves. *Dokl. Acad. Nauk SSSR* (3) **318**, 556 (in Russian).

PACKHAM, B. A. 1952 The theory of symmetrical gravity waves of finite amplitude II. The solitary wave. *Proc. R. Soc. Lond.* A **213**, 238.

PENNEL, S. A. 1987 On a series expansion for the solitary wave. *J. Fluid Mech.* **179**, 557.

PENNEL, S. A. & SU, C. H. 1984 A seventeenth-order series expansion for the solitary wave. *J. Fluid Mech.* **149**, 431.

PLOTNIKOV, P. I. 1983 Stokes conjecture proof in the theory of surface water waves. *Dokl. Acad. Nauk SSSR* (1) **269**, 80 (in Russian).

PLOTNIKOV, P. I. 1991 Nonuniqueness of solitary water waves and bifurcation theorem for critical points of smooth functionals. *Izv. AN SSSR, Matem.* (2) **55**, 339 (in Russian).

RAYLEIGH, LORD 1876 On waves. *Phil. Mag.* (5) **1**, 257.

RICHARDSON, A. R. 1920 Stationary waves in water. *Phil. Mag.* (6) **256**, 97. N 235.

RUSSEL, J. S. 1838 Report of the Committee on waves. *Rep Brit. Assn Adv. Sci., 1837*, p. 417.

STOKES, G. G. 1880 On the theory of oscillatory waves. *Mathematical and Physical Papers*, vol. 1, pp. 197, 314.

TOLAND, J. F. 1978 On the existence of a wave of greatest height and Stokes conjecture. *Proc. R. Soc. Lond.* A **363**, 469.

VILLAT, H. 1915 Sur l'écoulement des fluides resant. *Ann. Sci. École Norm. supér.* **32**.

WILLIAMS, J. M. 1981 Limiting gravity waves in water of finite depth. *Phil. Trans. R. Soc. Lond.* A **302**, 139.

WITTING, J. 1975 On the highest and other solitary waves. *SIAM J. Appl. Maths.* **28**, 700.

WITTING, J. 1981 High solitary waves in water: results of calculations. *NRL Rep.* 8505.